JOURNAL OF FUNCTIONAL ANALYSIS 22, 283–294 (1976)

Gleason Parts Separated by Smooth Curves

BERNT ØKSENDAL

Department of Mathematics, Agder Distriktshøgskole, Postboks 607, 4601 Kristiansand S., Norway

Communicated by John Werner

Received September 20, 1974; revised May 11, 1975

Suppose Γ is a simple closed C^2 curve in the complex plane and let W_1, W_2 be the components of the complement of Γ. Let X be a compact plane set. Necessary and sufficient conditions are given that any two points $x_1 \in X \cap W_1$ and $x_2 \in X \cap W_2$ belong to different Gleason parts for the algebra $R(X)$. We also give an answer to the question: How thin can a nontrivial part for $R(X)$ be?

1. INTRODUCTION

Let X be a compact subset of the complex plane $\mathbb{C}$ and let $R(X)$ be the uniform closure on X by the rational functions with poles outside X. Two points $x, y \in X$ are said to be in the same (Gleason) part for $R(X)$ if

$$\sup\{|f(y)|; f \in R(X), \|f\| \leq 1, f(x) = 0\} < 1.$$

Here $\|f\| = \sup\{|f(x)|; x \in X\}$. See [4] for properties of parts.

The first example of a set X which gives a disconnected part for $R(X)$ was discovered independent by A. M. Davie and J. Garnett. As is well known the example is a "string of beads" which consists of the closed unit disc with a disjoint sequence of open discs deleted, each having its center on the interval $(-1, 1)$. When the discs are properly chosen the points in the upper and lower part of the open unit disc are in the same part, while all the points on the boundary ∂X of X are peak points (and therefore one-point parts) for $R(X)$. (A point $x \in X$ is a peak point for $R(X)$ if there exists $f \in R(X)$ with $\|f\| = f(x) = 1$, $|f(y)| < 1$ for $y \neq x$). In this case we could say that the curve $[-1, 1]$ separates the two components of the nontrivial part for $R(X)$.

Now consider a more general situation: Γ is a simple, closed C^2 curve with complementary components W_1, W_2, and as before X is a compact set. When are any two points $x_1 \in X \cap W_1$ and $x_2 \in X \cap W_2$ in different parts? In Section 2 we give two answers, one in terms of analytic capacity and the other in terms of pointwise bounded approximation. The last characterization has some resemblance to the Forelli lemma for general uniform algebras (see [4, II.7.3]).

A well-known theorem by Browder [1] gives in particular that if P is a nontrivial part for $R(X)$ and $x \in P$, then P has full area density at x. Browder's theorem has subsequently been extended by Wang [10]. In Section 3 we give a different "bound" on how thin a nontrivial part P can be. Our result implies that if $x, y \in P$, $x \neq y$, then given any direction α almost all straight lines with that direction which passes between x and y, must meet P in a set of positive length.

2. Parts Separated by Smooth Curves

If K is a compact plane set we define the *analytic capacity* of K, $\gamma(K)$, by

$$\gamma(K) = \sup\{|f'(\infty)|; f \text{ analytic outside } K, f(\infty) = 0, |f(z)| \leqslant 1 \text{ for } z \notin K\},$$

and for general E we set

$$\gamma(E) = \sup\{\gamma(K); K \text{ compact}, K \subset E\}.$$

We list some properties of the set function γ, and refer the reader to [4] for the proofs and more information.

(i) γ is monotone, i.e., $\gamma(E) \leqslant \gamma(F)$ when $E \subset F$.

(ii) $\gamma(\Delta(x, r)) = r$, where $\Delta(x, r) = \{z; |z - x| < r\}$.

(iii) $\gamma(E)^2 \geqslant 1/\pi \cdot \text{Area}(E)$ for every Borel measurable set E.

(iv) If K is compact and connected then $\gamma(K) \geqslant \frac{1}{4} \cdot \text{diameter}(K)$.

$R(X)^{\perp}$ will be the set of (complex Borel) measures μ on X such that $\int f\, d\mu = 0$ for all $f \in R(X)$. If ν is a complex Borel measure, we will use the notation

$$\tilde{\nu}(\zeta) = \int (d\,|\nu|\,(z)/|z - \zeta|), \qquad \hat{\nu}(\zeta) = \int (d\nu(z)/z - \zeta) \text{ whenever defined.}$$

Using Fubini we see that $\tilde{\nu}(\zeta) < \infty$ a.e. $(dx\,dy)$. The following result is well-known (see [4, Proof of Theorem II.8.5]).

LEMMA 1. *Let $\mu \in R(X)^{\perp}$ and let ζ be a peak point for $R(X)$ such that $\tilde{\mu}(\zeta) < \infty$. Then $\hat{\mu}(\zeta) = 0$.*

Throughout the paper X will denote a compact subset of the complex plane $\mathbb{C}$. If $U \subset \mathbb{C}$ is open, then $H^{\infty}(U)$ will denote the set of bounded analytic functions on U. We will need the following result on a pointwise bounded approximation. The result stated is just a variation of well-known results due to Gamelin, Garnett, and Davie. Following Gamelin and Garnett [5] we define a *curvilinear null set* as a bounded subset of zero length of a C^2 curve. A *σ-curvilinear null set* will be a countable union of curvilinear null sets. Q will denote the set of nonpeak points for $R(X)$.

THEOREM 1. *Let U be an open set, and assume Area $(Q\backslash U) = 0$. Then the following are equivalent:*

(i) *For all $f \in H^{\infty}(U)$ there exists a sequence $\{f_n\} \subset R(X)$ with $\sup_n \|f_n\| < \infty$ and $f_n(x) \to f(x)$ for all $x \in U \cap X$.*

(ii) *For all $f \in H^{\infty}(U)$ there exists a sequence $\{f_n\} \subset R(X)$ with $\|f_n\| \leqslant \|f\|$ and $f_n(x) \to f(x)$ for all $x \in U \cap X$.*

(iii) *$\gamma(D\backslash U) \leqslant \gamma(D\backslash X)$ for each bounded open set D.*

(iv) *There is a σ-curvilinear null set E such that for each $z \in \partial U\backslash E$ there exists $r \geqslant 1$ satisfying*

$$\liminf_{\delta \to 0} \frac{\gamma(\Delta(z, r\delta)\backslash X)}{\gamma(\Delta(z, \delta) \cap \partial U)} > 0.$$

Proof. That (i), (iii), and (iv) are equivalent is really just the analog of [5, Theorem 10.5] for $R(X)$. The condition that Area$(Q\backslash U) = 0$ ensures that the proof of Theorem 10.5 goes through for $R(X)$ with straightforward modifications. Such a condition is needed in the modified proof of Lemma 10.2, where we need to know that the function

$$H(\zeta) = \frac{1}{\pi} \iint_{X\backslash U} \frac{h(z)}{z - \zeta} \cdot \frac{\partial g}{\partial \bar{z}} \, dx \, dy,$$

where $h \in L^{\infty}(dx\, dy)$, g is a C^{∞} function and $z = x + iy$, belongs to $R(X)$: Since Area$(Q\backslash U) = 0$, we can write

$$H(\zeta) = \frac{1}{\pi} \iint_{\partial\backslash U} \frac{h(z)}{z - \zeta} \cdot \frac{\partial g}{\partial \bar{z}} \, dx \, dy,$$

where ∂ is the set of peak points for $R(X)$.

Now let $\mu \in R(X)^{\perp}$. Then

$$\pi \int_X H(\zeta)\, d\mu(\zeta) = \iint_{\partial \setminus U} \int_X \frac{\partial g}{\partial \bar{z}} \cdot \frac{h(z)}{z-\zeta} \cdot d\mu(\zeta)\, dx\, dy$$

$$= \iint_{\partial \setminus U} \frac{\partial g}{\partial \bar{z}} \cdot h(z) \left(\int_X \frac{1}{z-\zeta}\, d\mu(\zeta) \right)$$

by Fubini, since the integrals converge absolutely. By Lemma 1 the inner integral is zero a.e. ($dx\, dy$) on ∂. So $\int H\, d\mu = 0$ and therefore $H \in R(X)$.

The rest of the proof that (i), (iii), and (iv) are equivalent goes as in [5] and is omitted.

Finally, the implication (i) $\Rightarrow$ (ii) follows by a variation of Davie's theorem in [2], as outlined by Gamelin and Garnett in [5, Lemma 10.1].

We will also need the following result, which can be found in [8, Theorem 2.3]. A *null set* E for $R(X)^{\perp}$ is a set which satisfies $|\mu|(E) = 0$ for all $\mu \in R(X)^{\perp}$.

THEOREM 2. *Let Γ be a simple, closed C^2 curve and let W be one of the components of $\mathbb{C}\setminus\Gamma$. Suppose $X \subset \overline{W}$. Then a (measurable) subset $E \subset X \cap \Gamma$ is a null set for $R(X)^{\perp}$ if and only if*

$$\liminf_{\delta \to 0} \frac{\gamma(\Delta(x, \delta) \cap W \setminus X)}{\delta} > 0$$

for a.a. $x \in E$ *with respect to arc length on* Γ.

THEOREM 3. *Let Γ be a simple, closed C^2 curve and let W_1, W_2 be the components of $\mathbb{C}\setminus\Gamma$. Then the following are equivalent:*

(i) *For all $f \in H^{\infty}(W_1 \cup W_2)$ there exists a sequence $\{f_n\} \subset R(X)$ such that $\| f_n \| \leqslant \| f \|$ and $f_n(x) \to f(x)$ for all $x \in X\setminus\Gamma$.*

(ii) *Any two points $x_1 \in W_1 \cap X$ and $x_2 \in W_2 \cap X$ are in different parts.*

(iii) $\liminf_{\delta \to 0} \gamma(\Delta(x, \delta)\setminus X)/\delta > 0$ *for* a.a. $x \in \Gamma$ *with respect to arc length.*

(iv) *For all* $\delta < \operatorname{diam}(\Gamma)$ *and for all* $x \in \Gamma$

$$\gamma(\Delta(x, \delta)\setminus X) \geqslant \tfrac{1}{4}\delta.$$

Proof. Choosing $f = 0$ on W_1, $f = 1$ on W_2 we see that (i) $\Rightarrow$ (ii). Moreover, using Theorem 1 with $U = W_1 \cup W_2$, we see that (i), (iii), and (iv) are equivalent. So it remains to prove that (ii) $\Rightarrow$ (iii): Put

$$E = \left\{ x \in \Gamma;\ \liminf_{\delta \to 0} \frac{\gamma(\Delta(x, \delta) \cap W_1 \backslash X)}{\delta} > 0 \right\},$$

$$F = \left\{ x \in \Gamma;\ \liminf_{\delta \to 0} \frac{\gamma(\Delta(x, \delta) \cap W_2 \backslash X)}{\delta} > 0 \right\}.$$

Let $H = \Gamma \backslash (E \cup F)$ and assume $ds(H) > 0$, where ds denotes arc length on Γ. By Theorem 2 there exists a measure $\nu \in R(X \cap \overline{W}_1)^{\perp}$ such that $|\nu|(H) > 0$, since $H \cap E = \varnothing$. By Glicksberg decomposition theorem ([4, Theorem II.7.11]) and the fact that $R(K)$ has no nonzero completely singular orthogonal measures ([11]), there exists a representing measure λ for a point $z_1 \in W_1 \cap X$ with respect to $R(\overline{W}_1 \cap X)$, i.e., a positive measure satisfying $\int f\, d\lambda = f(z_1)$ for all $f \in R(\overline{W}_1 \cap X)$, such that $\lambda(H) > 0$. Then of course λ is also a representing measure with respect to $R(X)$. Now assume (ii). Then by [7, Corollary 1.3] there exists a Borel set G such that $\lambda(G) = 1$ and $\rho(G) = 0$ for all representing measures ρ for $y \in W_2 \cap X$ with respect to $R(\overline{W}_2 \cap X)$. Since (by [11, Theorem 3.3]) any representing measure for a point y belonging to a part P is carried by $\bar{P}$ and represents y with respect to $R(\bar{P})$, we get that $\beta(G) = 0$, and therefore also $\beta(G \cap H) = 0$, for all representing measures β for points $y \in W_2 \cap X$ with respect to $R(\overline{W}_2 \cap X)$. Again by the Wilken result and the Glicksberg decomposition theorem, we get that $G \cap H$ is a null set for $R(\overline{W}_2 \cap X)^{\perp}$. On the other hand, since $\lambda(G \cap H) > 0$, we get by Theorem 2 that $ds(G \cap H) > 0$. These two last facts imply, again by Theorem 2, that there exist points $x \in H \cap G$ such that

$$\liminf_{\delta \to 0} \frac{\gamma(\Delta(x, \delta) \cap W_2 \backslash X)}{\delta} > 0.$$

This means that $x \in H \cap G \cap F$, a contradiction.

Note. Our proof of the equivalence of (i) and (ii) in Theorem 3 rests heavily on the use of analytic capacity. It would be interesting to see if a more direct proof of this result can be given. Another question is how crucial the condition that Γ is a C^2 arc is for this equivalence.

3. How Thin Can a Part Be?

As before we let $X \subset \mathbb{C}$ be compact. Fix a point $z_0 \in \mathbb{C}$. For $r \geqslant 0$ put

$$\Gamma_r = \{z; \, | z - z_0 | = r\}.$$

Let $a = \inf\{r; \Gamma_r \cap X \neq \varnothing\}$, $b = \sup\{r; \Gamma_r \cap X \neq \varnothing\}$ and define $M = \{r; d\theta(\Gamma_r \cap X \backslash \partial) = 0, a \leqslant r \leqslant b\}$, where ∂ as before is the set of peak points for $R(X)$ and $d\theta$ is linear measure on Γ_r. How thick can M be without forcing X to consist of more than one part? Before we give an answer to this unprecise question we recall the definition of logarithmic capacity, here denoted by cap:

If $K \subset \mathbb{C}$ is compact and β is a probability measure on K we define

$$U_\beta(z) = \int_K \log \left| \frac{1}{z - \zeta} \right| d\beta(\zeta),$$

the logarithmic potential of β. U_β is superharmonic and lower semicontinuous in $\mathbb{C}$. (See [9, II.23]).

Define

$$V = V(K) = \inf_\beta \sup_{C \backslash K} U_\beta(z)$$

where the inf is taken over all such measures β. The logarithmic capacity of K, cap(K), is then defined as

$$\operatorname{cap}(K) = e^{-V}.$$

The definition is extended to arbitrary sets E by

$$\operatorname{cap}(E) = \sup\{\operatorname{cap} K; K \subset E, K \text{ compact}\}.$$

Lemma 2. *Let μ be a complex measure. Then for all $r \geqslant 0$, except on a set Z of logarithmic capacity zero, we have*

$$\int_{\Gamma_r} \tilde{\mu}(z) \, d\, | z | < \infty.$$

Proof. Suppose not. Then we can find a compact set $K \subset [0, \infty\rangle$ with $\operatorname{cap}(K) > 0$ and such that

$$\int_{\Gamma_r} \tilde{\mu}(z) \, d\, | z | = \infty \qquad \text{for all} \quad r \in K. \tag{1}$$

Since $\operatorname{cap}(K) > 0$ we have $V(K) < \infty$ so we can find a probability measure β on K such that $\sup_{C \backslash K} U_\beta(z) \leqslant M < \infty$, where M is a constant. Since $U_\beta(z)$ is lower semicontinuous, we conclude that $U_\beta(z) \leqslant M$ everywhere.

By (1),

$$\int_K \left(\int_{\Gamma_r} \tilde{\mu}(z)\, d\,|\,z\,| \right) d\beta(r) = \infty.$$

So by Fubini we must also have

$$\int \left(\int_K \left(\int_{\Gamma_r} \frac{1}{|\,\zeta - z\,|}\, d\,|\,z\,| \right) d\beta(r) \right) d\,|\,\mu\,|\,(\zeta) = \infty.$$

Therefore

$$\int \left(\int_K \log \left| \frac{1}{r - |\,\zeta\,|} \right| d\beta(r) \right) d\,|\,\mu\,|\,(\zeta) = \infty,$$

i.e.,

$$\int U_\beta(|\,\zeta\,|)\, d\,|\,\mu\,|\,(\zeta) = \infty.$$

But $\int U_\beta(|\,\zeta\,|)\, d\,|\,\mu\,|(\zeta) \leqslant M \cdot \|\,\mu\,\| < \infty$, a contradiction.

See [6] for more information on the convergence of $\tilde{\mu}(z)$, the Newtonian potential of $|\,\mu\,|$.

LEMMA 3. *Let H be a subset of the real line and suppose* cap $H > 0$. *Then for all $x \in H$ except on a set of logarithmic capacity zero we have*

$$\operatorname{cap}(\langle x - r, x] \cap H) > 0 \qquad \text{and} \qquad \operatorname{cap}([x, x + r\rangle \cap H) > 0, \qquad \text{for all} \quad r > 0.$$

In other words, cap—almost all points of H are points of both left and right cap—density with respect to H.

Proof. The countable union of sets of logarithmic capacity zero still has logarithmic capacity zero ([9, III.8]). So it is enough to prove that cap-almost all points of H satisfies

$$\operatorname{cap}([x, x + r\rangle \cap H) > 0 \qquad \text{for all} \quad r > 0.$$

(The cap-left density is proved similarly.) Suppose this is not the case. Then there exists a subset E of H with cap $E > 0$ such that for all $x \in E$

$$\operatorname{cap}([x, x + r_x\rangle \cap H) = 0 \qquad \text{for some} \quad r_x > 0.$$

Then

$$E \subset \bigcup_{x \in E} [x, x + r_x\rangle \cap H.$$

Let F be the set of points $y \in E$ such that for all $x \in E$ we have $y \notin \langle x, r_x \rangle$. Then the collection $\{[y, y + r_y\rangle; y \in F\}$ is disjoint and therefore countable. Hence F is countable. But

$$E \subset \bigcup_{y \in F} [y, y + r_y\rangle \cap H \cup \bigcup_{x \in E \setminus F} \langle x, x + r_x \rangle \cap H.$$

The last union can be reduced to a countable union by Lindeløf's theorem, so we can write

$$E \subset \bigcup_{i=1}^{\infty} [x_i, x_i + r_i\rangle \cap H, \qquad \text{where} \quad x_i \in E \quad \text{and} \quad r_i = r_{x_i}.$$

Since $\operatorname{cap}([x_i, x_i + r_i \cap H) = 0$, we conclude that $\operatorname{cap} E = 0$, which is a contradiction.

A closed subset F of X is called a *peak set* for $R(X)$ if there exists a function $f \in R(X)$ such that $f(x) = 1$ for $x \in F$ and $|f(x)| < 1$ for $x \in X \setminus F$. Glicksberg's peak set theorem states that F is a peak set if and only if whenever $\mu \in R(X)^{\perp}$ we also have $\mu \mid F \in R(X)^{\perp}$. ([4, Theorem II.12.7]).

We let Γ_r and M be as before. If $\operatorname{cap}(M) > 0$ we let $M_0 \subset M$ denote the set of points of both left and right cap-density with respect to M.

Then $\operatorname{cap}(M \setminus M_0) = 0$ by Lemma 3. With this notation we now give an answer to the vague question asked in the beginning of this section:

THEOREM 4. *Suppose* $\operatorname{cap}(M) > 0$. *Then for every* $r \in M_0$ *the sets*

$$\overline{\Delta(z_0, r)} \cap X \qquad \text{and} \qquad X \setminus \overline{\Delta(z_0, r)}$$

are both peak sets for $R(X)$.

So, in particular, points in $\Delta(z_0, r) \cap X$ and $X \setminus \Delta(z_0, r)$ cannot be in the same part.

Proof. Choose $\mu \in R(X)^{\perp}$. By Lemma 2 we have

$$\int_{\Gamma_r} \tilde{\mu}(z) \, d \mid z \mid < \infty$$

for all $r \in M_\mu = M \setminus Z_\mu$, where $\operatorname{cap}(Z_\mu) = 0$. Since

$$\int_{\Gamma_r} \frac{d \mid z \mid}{\mid z - \zeta \mid} = \infty \qquad \text{for all} \quad \zeta \in \Gamma_r,$$

we must have $|\mu|(\Gamma_r) = 0$ for all $r \in M_\mu$. Moreover, since

$$\int_{\Gamma_r} \tilde{\mu}(z)\, d\,|\,z\,| < \infty,$$

we must have $\tilde{\mu}(z) < \infty$ a.e. $d\,|\,z\,|$ on Γ_r, $r \in M_\mu$. By Lemma 1 and the definition of M

$$\hat{\mu}(z) = 0 \qquad \text{a.e.} \quad d\,|\,z\,| \quad \text{on} \quad \Gamma_r, \quad r \in M_\mu .$$

Therefore, by Fubini,

$$0 = \int_{\Gamma_r} \hat{\mu}(z)\, dz = \int_{\zeta \in X} \left(\int_{\Gamma_r} \frac{dz}{\zeta - z} \right) d\mu(\zeta)$$

$$= \int_{\zeta \in X \setminus \Gamma_r} \left(\int_{\Gamma_r} \frac{dz}{\zeta - z} \right) d\mu(\zeta) = -2\pi i \cdot \mu(\Delta(z_0\,, r) \cap X).$$

So $\mu(\Delta(z_0\,, r) \cap X) = \mu(\overline{\Delta(z_0\,, r)} \cap X) = 0$ for $r \in M_\mu$.

Now fix $r \in M_0$. Then for all n we have $\operatorname{cap}([r, r + (1/n)] \cap M) > 0$ (Lemma 3), so we can find a sequence $\{r_n\}$ with $r_n \in M_\mu$ and $r_n \searrow r$. This gives $\mu(\overline{\Delta(z_0\,, r)} \cap X) = \lim_n \mu(\Delta(z_0\,, r_n) \cap X) = 0$. Similarly we find a sequence $\{s_n\}$, with $s_n \in M_\mu$ and $s_n \nearrow r$, so that

$$\mu(\Delta(z_0\,, r) \cap X) = \lim_n \mu(\Delta(z_0\,, s_n) \cap X) = 0.$$

Hence $\mu(\overline{\Delta(z_0\,, r)} \cap X) = \mu(\Delta(z_0\,, r) \cap X) = 0$ for all $r \in M_0$. Now choose $f \in R(X)$. The argument above applied to $\mu_1 = f\mu \in R(X)^\perp$ gives that

$$\mu_1(\overline{\Delta(z_0\,, r)} \cap X) = 0, \qquad \text{i.e.,} \quad \int_{\overline{\Delta(z_0, r)}} f\, d\mu = 0$$

and similarly $\int_{X \setminus \Delta(z_0, r)} f\, d\mu = 0$. By Glicksberg's peak set theorem we are done.

If we apply Theorem 4 to parts, we get the following "bound" on how thin a part can be:

COROLLARY 1. *Let P be a part for $R(X)$. Suppose $x \neq y$, $x, y \in P$. Then for all r in the interval $[0, |\,x - y\,|]$, except on a set of logarithmic capacity zero, the circle*

$$\Gamma_r(x) = \{z;\ |\,z - x\,| = r\}$$

must intersect P in a set of positive arc length.

Proof. Let m be a representing measure for x with respect to $R(X)$. Choose a function $f \in R(X)$ vanishing at x. Then the measure $\mu = fm$ is orthogonal to $R(X)$. If ζ belongs to a part $Q \neq P$ and $\tilde{\mu}(\zeta) < \infty$, then we have $\hat{\mu}(\zeta) = 0$, for otherwise the measure

$$\nu(z) = \frac{1}{\hat{\mu}(\zeta)(z - \zeta)} \cdot \mu(z)$$

represents ζ and is not singular to m (see [11, Lemma 2.5]). Hence if $\zeta \notin P$ and $\tilde{\mu}(\zeta) < \infty$ then $\hat{\mu}(\zeta) = 0$.

Suppose there exists a set M with $\operatorname{cap}(M) > 0$ and $d\theta(\Gamma_r \cap P) = 0$ for all $r \in M$. Then repeating the argument in the proof of Theorem 4 we conclude that

$$|\mu|(\Gamma_r) = \mu(\Delta(x, r) \cap X) = \mu(X \backslash \Delta(x, r)) = 0 \qquad \text{for} \quad r \in M_0$$
$$\text{where} \quad \operatorname{cap} M_0 > 0. \tag{*}$$

Therefore

$$\int_{X \backslash \Delta(x,r)} f\, dm = 0 \qquad \text{for all} \quad f \in R(X) \text{ vanishing at } x.$$

So

$$\int_{X \backslash \Delta(x,r)} g(z)\, dm(z) = g(x) \cdot m(X \backslash \Delta(x, r)) \qquad \text{for all} \quad g \in R(X).$$

Suppose there exists a representing measure m_0 for x such that $a = m_0(X \backslash \Delta(x, r)) > 0$. Then we conclude that

$$|g(x)| \leqslant (1/a) \cdot \sup\{|g(z)|;\, z \in X \backslash \Delta(x, r)\}, \qquad g \in R(X).$$

This is only possible if $\Delta(x, r) \subset X^0$, for otherwise $R(X) \mid F$ would be uniformly dense in $R(F)$, by the Runge theorem, where $F = [X \backslash \Delta(x, r)] \cup \{x\}$. But then we can choose $m = (1/2\pi)\, d\theta$ along Γ_r as a representing measure for x, and we obtain a measure μ which contradicts (*). Therefore we must have

$$m(X \backslash \Delta(x, r)) = 0 \qquad \text{for all} \quad m \text{ representing } x.$$

A similar argument gives that

$$n(X \cap \Delta(x, r)) = 0 \qquad \text{for all measures } n \text{ representing } y.$$

In particular, m and n must be singular for all m and n, so x and y cannot belong to the same part P. This contradiction proves the corollary.

Instead of using circles we could of course have used straight lines with any given direction in Theorem 4. This will give the following version of Corollary 1:

COROLLARY 2. *Let P be a part for $R(X)$. Suppose $x \neq y$, $x, y \in P$. Let α denote any given direction. Let L_r^α denote the straight line with direction α passing between x and y with a distance r from x.*

Then for all r except on a set of logarithmic capacity zero, the line L_r^α meets P in a set of positive length.

These two corollaries say roughly that a part cannot be too thin along too many curves. The question how thin a part can be is closely related to the following conjecture, raised by [11]:

CONJECTURE (W). Let Q be a part for $R(X)$ and assume $x \in \bar{Q}\backslash Q$. Then x is a peak point. In other words, $P \cap \bar{Q} = \varnothing$ for all distinct parts P, Q.

We end this discussion by showing how Corollary 2 relates this conjecture to a problem about analytic capacity and projection:

CONJECTURE (C). Let E be a bounded set. Let α denote a direction and let E_α be the projection of E along the direction α onto a line orthogonal to α. Suppose that for all directions α the set E_α misses only a set of logarithmic capacity zero to fill out the whole interval between its end points (i.e., the projection E_α is "cap-almost" an interval). Then

$$\gamma(E) \geqslant c \cdot \operatorname{diam}(E),$$

for some universal constant c.

We show that (C) $\Rightarrow$ (W): Assume (W) is not true. Then there exist distinct parts P, Q and $z_0 \in P \cap \bar{Q}$. By a result of [3]:

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n \backslash P_0) < \infty,$$

where $P_0 = P \cup \partial$ and A_n is the annulus $\{z;\, 2^{-n-1} \leqslant |z - z_0| \leqslant 2^{-n}\}$. In particular

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n \cap Q) < \infty,$$

and this implies

$$\frac{\gamma(\Delta(z_0, \delta) \cap Q)}{\delta} \to 0.$$

Choose $\delta > 0$ so small that $\gamma(\Delta(z_0, \delta) \cap Q)/\delta < c$ and $\delta < \operatorname{diam} Q$, where c is the constant from the conjecture (C). Then by the conjecture (C) there exists a direction α such that the projection E_α of $E = Q \cap \Delta(z_0, \delta)$ misses a set of positive logarithmic capacity to fill out the whole interval. Then Theorem 4 applied to $R(\bar{Q})$ (with lines instead of circles) gives that Q is not a part for $R(\bar{Q})$. Therefore Q is not a part for $R(X)$, using [4, Theorem VI.1.1] and [11, Theorem 3.3]. This contradiction proves that (C) $\Rightarrow$ (W).

Acknowledgments

The author is very grateful to John Garnett and Arne Stray for valuable conversations. He is indebted to John Garnett for pointing out a strengthening of Theorem 4.

References

1. A. Browder, Point derivations on function algebras, *J. Functional Analysis* **1** (1967), 22–27.
2. A. M. Davie, Bounded approximation and Dirichlet sets, *J. Functional Analysis* **6** (1970), 460–467.
3. A. M. Davie and D. Wilken, An extention of Melnikov's theorem, *J. Functional Analysis* **11** (1972), 179–183.
4. T. W. Gamelin, "Uniform Algebras," Prentice-Hall, Englewood Cliffs, N.J., 1969.
5. T. W. Gamelin and J. Garnett, Pointwise bounded approximation and Dirichlet algebras, *J. Functional Analysis* **8** (1971), 360–404.
6. J. Garnett, Analytic capacity and measure, Springer Lecture Notes in Math., Vol. 297.
7. I. Glicksberg, The abstract F. and M. Riesz theorem, *J. Functional Analysis* **1** (1967), 109–122.
8. B. Øksendal, Null sets for measures orthogonal to $R(X)$, *Amer. J. Math.* **94** (1972), 331–342.
9. M. Tsuji, Potential theory in modern function theory, Maruzen, 1959.
10. J. L. Wang, An approximate Taylor's theorem for $R(X)$, preprint math. inst., University of Aarhus, Danmark.
11. D. Wilken, The support of representing measures for $R(X)$, *Pacific J. Math.* **26** (1968), 621–626.